

Infrared behavior of the running coupling in scalar field theory

Marco Frasca*
Via Erasmo Gattamelata, 3
00176 Roma (Italy)
(Dated: May 7, 2010)

We compute the Green function of the massless scalar field theory in the infrared. Applying Callan-Symanzik equation we obtain the exact running coupling for this case by computing the beta function. This result is applied using a recently proved mapping theorem between a massless scalar field theory and Yang-Mills theory. This beta function gives a running coupling going to zero as p^4 in agreement with lattice results presented in Boucaud et al. [JHEP 0304 (2003) 005] and showing that the right definition of the running coupling for a Yang-Mills theory in the infrared is given in the MOM scheme.

PACS numbers: 11.90.+t, 11.15.Me, 11.15.-q, 11.10.Jj

I. INTRODUCTION

The study of a quantum field theory in a regime where the coupling becomes increasingly large appears to cope with a lot of difficulties. The problem relies on the absence of useful methods in this regime. This means that the only reliable approach used so far has been lattice computation.

The model we consider in this paper is a massless scalar field theory. Its relevance relies on the fact that Yang-Mills equations of motion can be reduced to the equation of motion of this theory giving solution for them in the infrared. This permits to obtain the Green function for the gluon [1, 2] and the behavior of the beta function using Callan-Symanzik equations.

For the gluon propagator there is a lot of research activity about, mostly trying to obtain its behavior by solving quantum Yang-Mills theory on the lattice [4–6]. The scenario that is going to emerge shows that the gluon propagator goes to a finite value when the momentum goes to zero. On the contrary, the running coupling goes to zero and against the common belief has no fixed point in the infrared.

The definition used in lattice computation has been given by Alkofer and von Smekal in [3]. This definition is quite satisfactory for a qualitative behavior but could be somewhat in disagreement with the real behavior. Indeed, this has been seen in a paper by Boucaud et al. [7] that show a different definition for the running coupling using the so called MOM scheme and compute its behavior on the lattice obtaining that this goes to zero as p^4 . Alkofer and von Smekal expects for the running coupling a fixed point with their definition but this can also goes to zero with the expected exponent quite different from the one by Boucaud et al. This opens a question about what should be the proper definition for this physical quantity.

An important answer has been given recently through the analysis of experimental data by a series of papers by Prosperi's group [8–10]. These authors take the meson spectrum and get the running coupling by a Bethe-Salpeter like formalism. The comparison with analytical methods shows that the experimental data tend to decrease toward zero lying well below an expected fixed point in the infrared.

The current definition of the running coupling, as given in [3], uses the dressing functions of the gluon and ghost propagator. One has for the gluon propagator

$$D(p) = \frac{Z(p)}{p^2}, \quad (1)$$

for the ghost propagator

$$D^G(p) = \frac{J(p)}{p^2}, \quad (2)$$

and the running coupling is

$$\alpha(p) = J^2(p)Z(p). \quad (3)$$

*marcofrasca@mclink.it

We will show consistently that the running coupling to be considered is not this one but rather the one discussed in [7] with a MOM scheme and going to zero in the infrared as p^4 .

The paper is structured as follows. In sec. II we derive the propagator of the massless scalar field theory and show when it solves Yang-Mills equations. In sec. III we compare this propagator with lattice results and numerical solution of the Dyson-Schwinger equations. In sec. IV we see show that our propagator solves Callan-Symanzyk equations giving the corresponding beta function. In this way we prove that the running coupling goes to zero in the infrared as p^4 . Finally, in sec. V conclusions are given.

II. TWO-POINT FUNCTIONS FOR MASSLESS SCALAR FIELD AND YANG-MILLS THEORY

Quantum field theory of a massless scalar field theory is formulated with the partition function

$$Z[j] = \int [d\phi] \exp \left[i \int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4 + j\phi \right) \right]. \quad (4)$$

Our approach is to consider the limit $\lambda \rightarrow \infty$. This limit has been considered before [11–19]. These authors have taken $\frac{1}{2}(\partial\phi)^2$ as a perturbation. This choice has the effect to produce a real singular series needing a proper regularization with no real chance to get finite results in the given limit. The reason for this relies on the fact that some dynamics must be allowed in this limit. This can be obtained by a gradient expansion [1, 2, 20]. A gradient expansion is obtained by rewriting the above functional as

$$Z[j] = \int [d\phi] \exp \left[i \int d^4x \left(\frac{1}{2}\dot{\phi}^2 - \frac{\lambda}{4}\phi^4 + j\phi \right) \right] \exp \left[-i \int d^4x (\nabla\phi)^2 \right]. \quad (5)$$

and considering as a perturbation the gradient part. Now, we note that, in the limit $\lambda \rightarrow \infty$, if a dynamics is allowed, the term $\dot{\phi}^2$ must be of the same order of the term $\lambda\phi^4$. So, we can safely rescale time as $t \rightarrow \sqrt{\lambda}t$ and rewrite the above functional as

$$Z[j] = \int [d\phi] \exp \left[i\sqrt{\lambda} \int d^4x \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{4}\phi^4 + j\phi \right) \right] \exp \left[-i\frac{1}{\sqrt{\lambda}} \int d^4x (\nabla\phi)^2 \right]. \quad (6)$$

and we see that is just a proof of the fact that the limit $\lambda \rightarrow \infty$ recover a gradient expansion. We need to properly rescale the current j to maintain the right ordering but this is just an arbitrary function. The consequence of this rescaling of time gives us immediately the conclusion that the strong coupling limit is a semiclassical limit [21]. So, we take $\phi = \bar{\phi} + \delta\phi$ being

$$\ddot{\bar{\phi}} + \bar{\phi}^3 = j. \quad (7)$$

In the infrared, with the energy going to zero, the following causal approximation does hold [22, 23]

$$\phi(t) \approx \int_0^t dt' G(t-t') j(t') \quad (8)$$

being

$$\ddot{G}(t) + G(t)^3 = \delta(t). \quad (9)$$

This is a small time approximation and has been recently recovered in a study of nonlinear waves [24]. It is somewhat surprising the proper working of Green functions in such nonlinear systems but this turns out useful to maintain all the machinery of quantum field theory. This means that the leading order approximation to hold for our functional in the infrared limit is

$$Z[j] = \exp \left[-\frac{i}{2} \int d^4x_1 d^4x_2 \frac{\delta}{\delta j(x_1)} \nabla^2 \delta^3(x_1 - x_2) \frac{\delta}{\delta j(x_2)} \right] \exp \left[\frac{i}{2} \int d^4x_1 d^4x_2 j(x_1) \Delta(x_1 - x_2) j(x_2) \right] \quad (10)$$

being

$$\Delta(x_1 - x_2) = \delta^3(x_1 - x_2) [\theta(t_1 - t_2) G(t_1 - t_2) + \theta(t_2 - t_1) G(t_2 - t_1)] \quad (11)$$

and

$$G(t) = \theta(t)\mu \left(\frac{2}{\lambda}\right)^{\frac{1}{4}} \text{sn} \left[\left(\frac{\lambda}{2}\right)^{\frac{1}{4}} \mu t, i \right] \quad (12)$$

being μ an integration constant having the dimension of an energy. So, we showed that a massless scalar theory in the infrared take an integrable form. Using the series [29]

$$\text{sn}(u, i) = \frac{2\pi}{K(i)} \sum_{n=0}^{\infty} \frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}} \sin \left[(2n+1) \frac{\pi u}{2K(i)} \right] \quad (13)$$

being $K(i)$ the constant

$$K(i) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + \sin^2 \theta}} \approx 1.3111028777. \quad (14)$$

and after a Fourier transform, we get the full propagator

$$G(p) = \sum_{n=0}^{\infty} \frac{B_n}{p^2 - \omega_n^2 + i\epsilon} \quad (15)$$

being

$$B_n = (2n+1) \frac{\pi^2}{K^2(i)} \frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}}, \quad (16)$$

and

$$\omega_n = \left(n + \frac{1}{2}\right) \frac{\pi}{K(i)} \left(\frac{\lambda}{2}\right)^{\frac{1}{4}} \mu. \quad (17)$$

From this analysis we see that the theory, in the infrared limit, is trivial having a generating functional as a free theory while the propagator is in agreement with the Källén-Lehman representation [1, 30]. We also note the scaling of the propagator with $\lambda^{\frac{1}{4}}$ that already says to us that the coupling should go to zero as p^4 in a renormalization group analysis. We will discuss this in sec.IV.

This approach is rather general and gives a theoretical framework to treat a quantum field theory in the infrared limit. So, an important step is its application to a pure Yang-Mills theory. As already said, there is a lot of activity about the solution of this theory in the infrared mostly because there is a serious interpretation problem for the spectrum of the light unflavored mesons. For a Yang-Mills theory we will have to solve the equation [31]

$$\partial^\mu \partial_\mu A_\nu^a - \partial_\nu (\partial^\mu A_\mu^a) + g f^{abc} A^{b\mu} (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) + g f^{abc} \partial^\mu (A_\mu^b A_\nu^c) + g^2 f^{abc} f^{cde} A^{b\mu} A_\mu^d A_\nu^e = j_\nu^a. \quad (18)$$

being f^{abc} the structure constants of the Lie group and g the coupling constant. When we try to solve for a gradient expansion these equations we meet a severe problem. Most solutions are just chaotic [32–34] and are useless to build a quantum field theory. In order to get a set of solutions to start building a quantum field theory, we can apply a mapping theorem recently proved [25–27]

Theorem 1 (Mapping) *An extremum of the action*

$$S = \int d^4x \left[\frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{4} \phi^4 \right] \quad (19)$$

is also an extremum of the $SU(N)$ Yang-Mills Lagrangian when one properly chooses A_μ^a with some components being zero and all others being equal, and $\lambda = Ng^2$, being g the coupling constant of the Yang-Mills field, when only time dependence is retained. In the most general case the following mapping holds

$$A_\mu^a(x) = \eta_\mu^a \phi(x) + O(1/\sqrt{N}g) \quad (20)$$

being η_μ^a constant, that becomes exact for the Lorenz gauge.

Recent lattice computation in 2+1 dimensions have given strong support to this theorem [28]. So, with this choice, the equation for the Green function of a Yang-Mills theory, in a strong coupling limit, reduces to the one of the case of the massless scalar theory with the substitution $\lambda \rightarrow Ng^2$ for a SU(N) group and this is just the 't Hooft coupling as should be expected [2, 25]. So, finally, e.g. the gluon propagator in the Landau gauge can be written as

$$D_{\mu\nu}^{ab}(p) = \delta^{ab} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) G(p) + O(1/\sqrt{N}g) \quad (21)$$

being $G(p)$ given in eq.(15) with λ substituted by Ng^2 . This means that we are approximating the solution of eq.(18) with

$$A_\mu^a(x) \approx \int d^4x' D_{\mu\nu}^{ab}(x-x') j^{b\nu}(x') + O(1/\sqrt{N}g). \quad (22)$$

This latter equation should be checked on lattice computations.

It would not be difficult to prove that, with this class of solutions of Yang-Mills equations, the ghost propagator is the one of a free particle[25]:

$$D^G(p) = \frac{1}{p^2} + O(1/\sqrt{N}g). \quad (23)$$

giving in our case $J(p) = 1$ for its dressing function. We can similarly get for the dressing function of the gluon propagator, $Z(p) = G(p)p^2$. We see that when p goes to zero then we have

$$Z(p) \approx G(0)p^2 \quad (24)$$

with $G(0) \neq 0$ as can be immediately realized. Then we get the infrared indices $k_1 = 0$ for the ghost and $k_2 = 1$ for the gluon clearly different from those given in [3, 35]. So one gets for the running coupling as defined by Alkofer and von Smekal $\alpha(p) = G(0)p^2$ that goes to zero in the infrared limit $p \rightarrow 0$. We then get a third index $k_3 = 2$ for the running coupling. The value of this third index will be the main point of our discussion in sec. IV.

III. NUMERICAL RESULTS

Having obtained an explicit expression for the gluon propagator we have to see how this expression compares with respect to numerical results. We will consider two kind of comparison for our aims: firstly we compare it with the most advanced lattice results and secondly we will consider a numerical solution of the Dyson-Schwinger equation.

The problem with Dyson-Schwinger equation is that a scenario has been proposed [3] enforcing the view that the gluon propagator should go to zero at small momenta, the ghost propagator should go to infinity faster than the free particle case and that the running coupling reaches a fixed point in the same limit. Recent data on lattice are showing that this view is not correct [4–6]. Besides, data extracted from meson spectrum also showed that the running coupling is not going to reach a fixed point bending clearly toward zero [8–10].

About numerical solutions of Dyson-Schwinger equation there exist two kind of solutions: on a compact manifold [36, 37] and D=3+1 [38]. For the solution on a compact manifold there are contradictory results but it is clearly seen the running coupling bending toward zero. We cannot use these data for comparison. But we can compare with the results due to Aguilar and Natale [38]. Solving Dyson-Schwinger equations in D=3+1 these authors were able to recover the scenario that is presently emerging from lattice computation. They get a gluon propagator going to a finite value at lower momenta, the ghost propagator converging to the free one and the running coupling as defined by Alkofer and von Smekal going to zero. Then, we must consider these as the proper reference data to compare.

An important point to be noticed is that there is only one parameter to be computed to compare our propagator with numerical data and this is the gluon mass that we write explicitly as $m_g = (\pi/2K(i))\mu(Ng^2/2)^{\frac{1}{4}}$. From this value one gets the integration constant μ that is a number to be decided experimentally.

Firstly, let us see as our propagator compare to the most up to date lattice data [4]. We see the results in fig.1. At lower momenta the agreement is perfect with a gluon mass of 545 MeV. There are volume effects but this are seen only in the intermediate energy region. This was also seen in [1]. It should be expected that increasing the lattice volume should make both curves coincide.

Finally, we compare our data with the numerical results of Aguilar and Natale [38]. In this case we must have complete coincidence and this is exactly what happens. This can be seen in fig.2. The fit is obtained with a gluon mass of 738 MeV. From this figure is clearly seen that Aguilar and Natale hit numerically our propagator. With a proper fit they could have obtained all the full glueball spectrum.

The agreements with lattice data and numerical solution are astonishingly good. So, the next step is to draw one more important consequence from our solution.

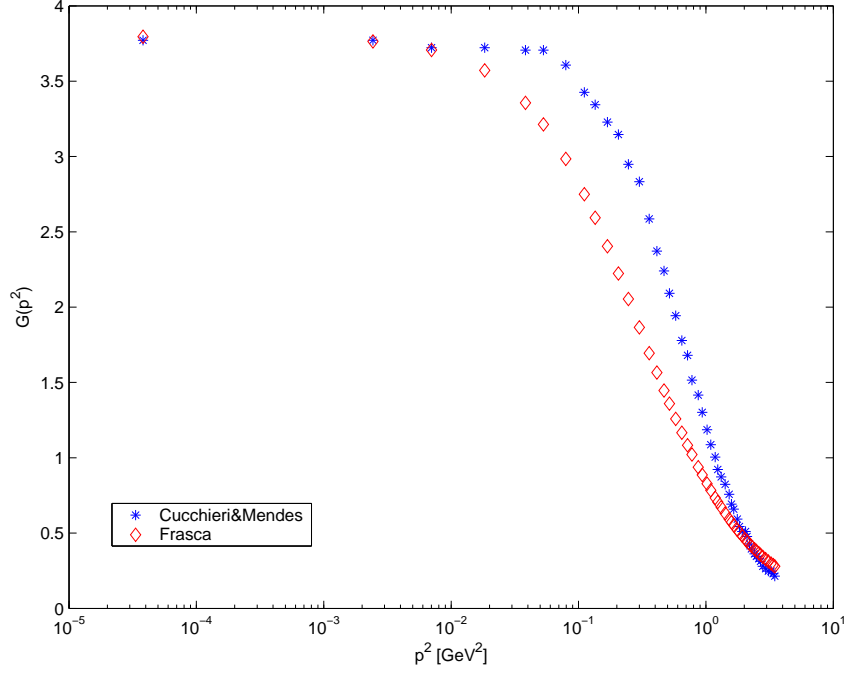


FIG. 1: Gluon propagator compared to lattice data in [4].

IV. CALLAN-SYMANZYK EQUATION AND RUNNING COUPLING

A proper definition of the running coupling in the infrared is not a trivial matter. An interesting analysis has been presented in [8]. Lattice computations use the generally accepted definition due to Alkofer and von Smekal [3]. It is a obvious matter that a proper definiton can be only obtained having the gluon propagator and solving the Callan-Symanzyk equation. This is exactly our aim. Let us point out that in [7] an analysis was carried out through lattice computation, assuming a MOM scheme to hold for the running coupling in the form

$$\alpha_{\text{MOM}}(p) \equiv \frac{g_R^2}{4\pi} = \frac{1}{4\pi} \left[\frac{G^{(3)}(p^2, p^2, p^2)}{(G^{(2)}(p^2))^3} (p^2 G^{(2)}(p^2))^{3/2} \right]^2 \quad (25)$$

being $G^{(3)}(p^2, p^2, p^2)$ the three-gluon Green function and $G^{(2)}(p^2)$ the two-point function corresponding to our $G(p)$. These authors prove that, on the lattice, $\alpha_{\text{MOM}}(p) \propto p^4$. We see that, if $G^{(2)}(0) = \text{constant}$, to have the running coupling going as p^4 implies $G^{(3)}(p^2, p^2, p^2) \propto 1/p$ and also this is in agreement with Boucaud et al. conclusions. Indeed, these authors have given a description of the infrared behavior of a pure Yang-Mills theory in close agreement with the one that is presently emerging from lattice computations (see e.g. [42] and refs. therein).

The propagator of the massless scalar theory we have derived above must satisfy the Callan-Symanzik equation

$$\mu \frac{\partial G(\mathbf{x}, t)}{\partial \mu} + \beta(\lambda) \frac{\partial G(\mathbf{x}, t)}{\partial \lambda} + 2\gamma G(\mathbf{x}, t) = 0 \quad (26)$$

and setting $\mathbf{x} = 0$ we have immediately

$$\mu \frac{\partial G(t)}{\partial \mu} + \beta(\lambda) \frac{\partial G(t)}{\partial \lambda} + 2\gamma G(t) = 0 \quad (27)$$

or, working with momentum, one has

$$\mu \frac{\partial G(p)}{\partial \mu} + \beta(\lambda) \frac{\partial G(p)}{\partial \lambda} + 2\gamma G(p) = 0. \quad (28)$$

We see immediately that this is true if

$$\beta(\lambda) = 4\lambda \quad (29)$$

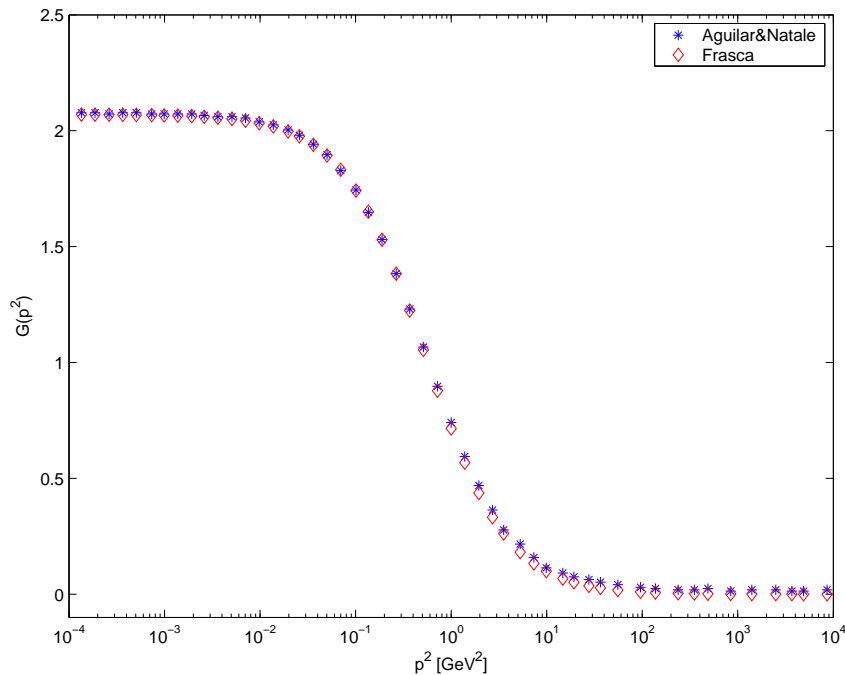


FIG. 2: Gluon propagator compared to numerical data in [38].

and

$$\gamma = -1 \quad (30)$$

giving us the behavior of the running coupling we looked for. This result is in agreement with recent analysis [39, 40] where such a form of beta function in the strong coupling limit was postulated. Similarly, from AdS/CFT correspondence a similar conclusion was drawn [41].

We know that

$$\beta(\lambda) = \mu \frac{d\lambda}{d\mu} \quad (31)$$

giving immediately

$$\lambda(p) = \lambda_0 \frac{p^4}{\mu^4}. \quad (32)$$

Now, with the substitution $\lambda \rightarrow Ng^2$ that maps the scalar theory on the Yang-Mills theory we get immediately the result given in [7]. This permits us to draw immediately the conclusion that the definition $\alpha_{MOM}(p^2)$ is the proper one for the running coupling in the infrared. This is expected to go to zero with a fourth power law reaching no fixed point. This gives index k_3 being 4. Such a result can be understood if we recall what we have said above about the ghost propagator. The ghost in the infrared limit decouples from the gluon and then, a definition of the running coupling implying the ghost propagator has no physical meaning.

Similarly, we have

$$\gamma = \frac{1}{2} \mu \frac{dZ}{d\mu} \quad (33)$$

being Z the renormalization constant of the field. We see immediately that $Z = p^2/\mu^2$ being this the expected scaling for the field. We just note that in the infrared limit particles are not gluons but rather the particles in the mass spectrum of the theory that should be properly called *glueballs*.

V. CONCLUSIONS

The conclusions to be drawn describe a scenario completely different from the one discussed in literature in these latter years. This is due mostly to lattice data that have shaken during these years the common beliefs that were going to form on known theoretical methods.

Presently, evidence is mounting that the gluon propagator reaches a finite value for momenta going to zero. In this paper we have shown that the proper definition of the running coupling in the infrared should be taken in a MOM scheme. This is shown to go to zero, not to a fixed point, with a fourth power of momentum. Finally, this result agrees well with the view that the ghost in the infrared decouples from the gluon and behaves as a free particle.

As this scenario is emerging from numerical solutions of the Yang-Mills quantum field theory and from phenomenological analysis, increasing confirmations in the future years have to be expected.

-
- [1] M. Frasca, Int. J. Mod. Phys. A **22**, 2433 (2007).
 - [2] M. Frasca, Int. J. Mod. Phys. A **22**, 1727 (2007).
 - [3] R. Alkofer, L. von Smekal, Phys. Rept. **353**, 281 (2001).
 - [4] A. Cucchieri, T. Mendes, PoS (LATTICE 2007) 297.
 - [5] I.L. Bogolubsky, E.M. Ilgenfritz, M. Müller-Preussker, A. Sternbeck, PoS (LATTICE 2007) 290.
 - [6] A. Sternbeck, L. von Smekal, D. B. Leinweber, A. G. Williams, PoS (LATTICE 2007) 340.
 - [7] Ph. Boucaud, F. De Soto, A. Le Yaouanc, J.P. Leroy, J. Micheli, H. Moutarde, O. Pène, J. Rodriguez-Quintero, JHEP 0304 (2003) 005.
 - [8] G. M. Prospero, M. Raciti, C. Simolo, Prog. Part. Nucl. Phys. **58**, 387 (2007).
 - [9] M. Baldicchi, A. V. Nesterenko, G. M. Prospero, D. V. Shirkov, C. Simolo, Phys. Rev. Lett. **99**, 242001 (2007).
 - [10] M. Baldicchi, A. V. Nesterenko, G. M. Prospero, C. Simolo, Phys. Rev. D **77**, 034013 (2008).
 - [11] S. Kovesi-Domokos, Nuovo Cim. A **33**, 769 (1976).
 - [12] R. Benzi, G. Martinelli, and G. Parisi, Nucl. Phys. B **135**, 429 (1978).
 - [13] C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, Phys. Rev. D **19**, 1865 (1979).
 - [14] N. Parga, D. Toussaint, and J. R. Fulco, Phys. Rev. D **20**, 887 (1979).
 - [15] C. M. Bender, F. Cooper, G. S. Guralnik, D. H. Sharp, R. Roskies, and M. L. Silverstein, Phys. Rev. D **20**, 1374 (1979).
 - [16] C. M. Bender, F. Cooper, G. S. Guralnik, H. Moreno, R. Roskies, and D. H. Sharp, Phys. Rev. Lett. **45**, 501 (1980).
 - [17] F. Cooper, and R. Kenway, Phys. Rev. D **24**, 2706 (1981).
 - [18] C. Bender, F. Cooper, R. Kenway, and L. M. Simmons, Phys. Rev. D **24**, 2693 (1981).
 - [19] N. F. Svaiter, Physica A **345**, 517 (2005).
 - [20] M. Frasca, Phys. Rev. D **73**, 027701 (2006); Erratum-ibid., 049902 (2006).
 - [21] M. Frasca, Proc. R. Soc. A **463**, 2195 (2007).
 - [22] M. Frasca, Mod. Phys. Lett. A **22**, 1293 (2007).
 - [23] M. Frasca, arXiv:0704.1568, to appear in International Journal of Modern Physics A.
 - [24] F. Dini, M. Molaie Emamzadeh, S. Khorasani, J. L. Bobin, R. Amrollahi, M. Sodagar, M. Khoshnagar, Phys. Scr. **77**, 025504 (2008).
 - [25] M. Frasca, Phys. Lett. **B670**, 73 (2008).
 - [26] M. Frasca, Mod. Phys. Lett. A **24**, 2425 (2009).
 - [27] T. Tao, private communication (2009).
 - [28] R. Frigori, Nucl. Phys. B **833**, 17 (2010).
 - [29] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, 2000).
 - [30] M. Frasca, arXiv:0711.3860 [hep-th].
 - [31] M. E. Peskin, D. V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus, Reading, 1995), Ch.12.
 - [32] S. G. Matinyan, G.K. Savvidy, N.G. Ter-Arutunian Savvidy, Sov. Phys. JETP **53**, 421 (1981).
 - [33] G. K. Savvidy, Phys. Lett. B **130**, 303 (1983).
 - [34] G. K. Savvidy, Nucl. Phys. B **246**, 302 (1984).
 - [35] E.-M. Ilgenfritz, M. Müller-Preussker, A. Sternbeck, A. Schiller, I. L. Bogolubsky, Braz. J. Phys. **37**, 193 (2007).
 - [36] C. S. Fischer, B. Gruter, R. Alkofer, Annals Phys. **321**, 1918 (2006).
 - [37] C. S. Fischer, A. Maas, J. M. Pawłowski, L. von Smekal, hep-ph/0701050.
 - [38] A. C. Aguilar, A. A. Natale, JHEP 0408 (2004) 057.
 - [39] I. M. Suslov, arXiv:0911.1149v1 [hep-th].
 - [40] I. M. Suslov, arXiv:0804.0368v3 [hep-ph].
 - [41] D. Podolsky, arXiv:1003.3670v1 [hep-th].
 - [42] Ph. Boucaud, Th. Brüntjen, J.P. Leroy, A. Le Yaouanc, A.Y. Lokhov, J. Micheli, O. Pène, J. Rodriguez-Quintero, C. Roiesnel, hep-lat/0602006.